MATH 161 - Fall 2013

LECTURE 4

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CALCULATING LIMITS

Limits Laws : Spse that c is a constant and the limits

$$\lim_{x\to a} f(x) \text{ and } \lim_{x\to a} g(x) \text{ exist.}$$

Then :

1) $$\lim_{x\to a} [f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$

2) $$\lim_{x\to a} [f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$$

3) $$\lim_{x\to a} [cf(x)] = c \cdot \lim_{x\to a} f(x)$$

4) $$\lim_{x\to a} [f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$$

5) $$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \quad \text{if } \lim_{x\to a} g(x) \neq 0$$

6) $\lim\limits_{x \to a} [f(x)]^n = \left[\lim\limits_{x \to a} f(x)\right]^n$

In order to apply these limit, we need to use two special limits

7) $\lim\limits_{x \to a} c = c$

8) $\lim\limits_{x \to a} x = a$

Then from Law 8 and Law 6

9) $\lim\limits_{x \to a} x^n = \left[\lim\limits_{x \to a} x\right]^n = a^n$

Similar result holds for roots as follows :

10) $\lim\limits_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$ where n is a positive integer

If n is even, we require that $a > 0$.

More generally,

11) $\lim\limits_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x \to a} f(x)}$ where n is a positive integer

If n is even, we require that $\lim\limits_{x \to a} f(x) > 0$

Ex 1

a) $\lim_{x \to 2} (2x^2 - 3x + 4) = \lim_{x \to 2} (2x^2) - \lim_{x \to 2} (3x) + \lim_{x \to 2} 4 \longrightarrow$ By 1 and 2

$= 2 \lim_{x \to 2} x^2 - 3 \lim_{x \to 2} x + \lim_{x \to 2} 4 \longrightarrow$ By 3

$= 2(2)^2 - 3 \cdot 2 + 4 \longrightarrow 9, 8, 7$

$= 8 - 6 + 4 = \underline{6}.$

b) $\lim_{x \to 2} \dfrac{x^3 + 2x + 1}{5 - 2x} = \dfrac{\lim_{x \to 2} (x^3 + 2x + 1)}{\lim_{x \to 2} (5 - 2x)} = \dfrac{\lim_{x \to 2} x^3 + 2 \lim_{x \to 2} x + \lim_{x \to 2} 1}{\lim_{x \to 2} 5 - 2 \lim_{x \to 2} x}$

$= \dfrac{8 + 4 + 1}{5 - 4} = \dfrac{13}{1} = \underline{13}$

Remark Plug in 2 to $2x^2 - 3x + 4$ and we get 6.

Plug in 2 to $\dfrac{x^3 + 2x + 1}{5 - 2x}$ and we get 13.

DIRECT SUBSTITUTION PROPERTY

If f is a polynomial or a rational function and a is in the domain of f

then $\lim_{x \to a} f(x) = f(a)$.

Ex $\lim_{x \to a} \cos(x) = \cos a$

$\lim_{x \to a} \sin(x) = \sin a$.

Ex Compute $\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$

We can find the limit by substitution because 1 is not in the domain.

However let's plug in 1 for x. We get $\frac{0}{0}$. In this case we will need to do some preliminary algebra.

$$\frac{x^2 - 1}{x - 1} = \frac{(x-1)(x+1)}{(x-1)}$$

Since we are trying to find limit as x approaches 1, we have $x \neq 1$ and so $x - 1 \neq 0$

Hence we can cancel the common factor and compute limits as follows:

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{(x-1)(x+1)}{(x-1)} = \lim_{x\to 1} (x+1) = 1+1 = 2$$

Rmk If $f(x) = g(x)$ when $x \neq a$, then $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$, provided the limit exists.

Ex $\lim_{h\to 0} \frac{(3+h)^2-9}{h}$. Plug in 0 and we get $\frac{0}{0}$

Again simplifying,

$$\lim_{h\to 0} \frac{9+6h+h^2-9}{h} = \lim_{h\to 0} \frac{6h+h^2}{h} = \lim_{h\to 0} \frac{\cancel{h}(6+h)}{\cancel{h}} = \lim_{h\to 0} 6+h = 6$$

Ex

$$\lim_{t \to 0} \frac{\sqrt{t^2+9} - 3}{t^2}$$. Again plug in we get $\frac{0}{0}$, so need to simplify

Here we need to rationalize the numerator

$$\lim_{t \to 0} \frac{\sqrt{t^2+9} - 3}{t^2} \cdot \frac{\sqrt{t^2+9} + 3}{\sqrt{t^2+9} + 3}$$

$$= \lim_{t \to 0} \frac{(t^2+9) - 9}{t^2 \left[\sqrt{t^2+9} + 3\right]} = \lim_{t \to 0} \frac{\cancel{t^2}}{\cancel{t^2} \left[\sqrt{t^2+9} + 3\right]}$$

$$= \lim_{t \to 0} \frac{1}{\sqrt{t^2+9} + 3} = \frac{1}{\sqrt{0^2+9} + 3} = \frac{1}{3+3} = \frac{1}{6}$$

Ex $\lim_{x \to 0} |x| = ?$

Recall

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Since $|x| = x$ for $x > 0$,

we have $\lim\limits_{x \to 0^+} |x| = \lim\limits_{x \to 0^+} x = 0$

Since, $|x| = -x$,

$$\lim_{x \to 0^-} |x| = \lim_{x \to 0^-} (-x) = 0$$

So $\lim\limits_{x \to 0} |x| = 0 \longrightarrow$ Since left handed and rt handed limit agree.

Ex $\lim\limits_{x \to 0} \dfrac{|x|}{x} = ?$

$$\lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} \frac{x}{x} = \lim_{x \to 0^+} 1 = 1$$

$$\lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} \frac{-x}{x} = \lim_{x \to 0^-} -1 = -1$$

Since left handed and right handed limit don't agree, limit DNE.

THE SQUEEZE THM

If $f(x) \leq g(x) \leq h(x)$ when x is near a (except possible at a)

and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$

Then $\lim_{x \to a} g(x) = L$

Ex Show that $\lim_{x \to 0} x^2 . \sin \frac{1}{x} = 0$.

We cannot use $\lim_{x \to 0} x^2 \sin \frac{1}{x} = \lim_{x \to 0} x^2 . \lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ because

$\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ does not exist.

Here we apply squeeze thm.

We Know that

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$$

$$\Downarrow$$

because $x^2 \geq 0$, we can multiply and not change inequality

$$-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2$$

Then, $\lim\limits_{x \to 0} x^2 = (0)^2 = 0$

$\lim\limits_{x \to 0} -(x^2) = -(0)^2 = 0$

Then by Squeeze Thm, $\lim\limits_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$

Ex $\boxed{\lim\limits_{x \to 0} \frac{\sin x}{x} = 1}$

Ex Find $\lim\limits_{x \to 0} \frac{\sin 7x}{4x}$

Write $\frac{\sin 7x}{4x} = \frac{7}{4}\left(\frac{\sin 7x}{7x}\right)$

Let $\theta = 7x$

Since as $x \longrightarrow 0$, we have $\theta \longrightarrow 0$ and

hence $\lim\limits_{x \to 0} \frac{\sin 7x}{7x} = \lim\limits_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$.

Hence, $\lim\limits_{x \to 0} \dfrac{\sin 7x}{4x} = \lim\limits_{x \to 0} \dfrac{7}{4} \cdot \dfrac{\sin 7x}{7x}$

$= \dfrac{7}{4} \lim\limits_{x \to 0} \dfrac{\sin 7x}{7x} = \dfrac{7}{4} \cdot 1 = \dfrac{7}{4}$